

## FEEDBACK SPREADING CONTROL UNDER SPEED CONSTRAINTS\*

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**Abstract.** This paper is concerned with the control of spread in semilinear parabolic systems. It first introduces a formula in order to measure the speed of a spread. Then feedback spreading control laws under speed constraints are studied by a set-valued approach. The optimality of such laws is examined in the case in which the system is affine dependent upon the control.

**Key words.** semilinear parabolic systems, spreading control, lower semicontinuity of maps, constrained optimization

**AMS subject classifications.** Primary, 93C20, 49J20; Secondary, 54C60

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**1. Introduction.** Spreadable distributed parameter systems provide a mathematical context for modeling expansion phenomena which may arise in spatially distributed processes; cf. [6, 7] and the references therein.

In handling the control aspects of that concept, a first attempt has been made in [8], where it is shown that spreading control can be determined by minimizing a rather unusual criterion, which is partly quadratic but contains a nonquadratic term. Conditions for a solution are given, the optimality system is derived, and algorithms for the resolution are determined. It is even of interest to cite [9], which relates spreading control to actuators for a class of linear distributed parameter systems.

Nevertheless, all of the approaches cited above have the disadvantage of being restricted to linear systems and concern only a few situations. In a recent study [13], it has been pointed out that feedback spreading controls for semilinear partial differential equations may be investigated in the framework of *monotone* solutions with respect to a *preorder*; cf. [1, 17]. Then the application of some results on monotonicity by [4] has allowed us to characterize these controls as selections of a certain set-valued map, which is defined by a set of tangential conditions.

The present study continues the investigation of the field as expounded in [13] by essentially concentrating on the speed of a spread. For this, we are motivated by the technical need to design spreads, taking into consideration both the speed and the time of spreading; cf. [8]. First, we propose a convenient setting in which the measure of the spread speed can be rigorously made. Then, due to some set-valued analysis facts, we examine the existence of feedback spreading control laws, which generate a spread either slower or quicker than a desired given speed.

In this paper, the following definitions and notation are used. Let  $Y$  be a Hilbert space; then a set-valued map  $Q : \mathcal{S} \rightarrow 2^Y \setminus \{\emptyset\}$  is said to be lower semicontinuous (lsc) whenever the following property holds: For each  $z_0 \in \mathcal{S}$  and any sequence of elements  $z_n \in \mathcal{S}$  converging to  $z_0$ , for every  $y_0 \in Q(z_0)$ , there exists a sequence of elements  $y_n \in Q(z_n)$  which converges to  $y_0$ .

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The graph of  $Q$  is denoted by

$$\text{graph}(Q) \doteq \{(z, y) \in \mathcal{S} \times Z \mid y \in Q(z)\}.$$

The inverse of  $Q$  is the map  $Q^{-1} : Y \rightarrow 2^{\mathcal{S}}$  defined by

$$Q^{-1}(y) \doteq \{z \in \mathcal{S} \mid y \in Q(z)\} \quad \text{for each } y \in Y.$$

A selection of the map  $Q$  is a mapping  $\nu : \mathcal{S} \rightarrow Y$ , which satisfies

$$\nu(z) \in Q(z) \quad \text{for each } z \in \mathcal{S}.$$

We quote Michael's selection theorem, which states that any lsc set-valued map with closed convex values has a continuous selection; cf. [5].

A mapping from  $Z$  to  $Y$  is said to be demicontinuous if it maps strongly convergent sequences in  $Z$  into weakly convergent sequences in  $Y$ ; cf. [15].

When the scalar product in  $Y$  is clear from the context, it is denoted by  $\langle \cdot ; \cdot \rangle$ .

The projector of best approximation on a closed convex subset  $\mathcal{K}$  of  $Y$  will be denoted by  $\pi_{\mathcal{K}}(\cdot)$ .

The directional derivative of a functional  $\ell : \mathcal{S} \rightarrow \mathbb{R}$  in the direction of  $y \in Y$ , if it exists at a point  $z \in \mathcal{S}$ , is denoted by

$$(1.1) \quad d\ell(z)(y) \doteq \liminf_{h \downarrow 0} \frac{\ell(z + hy) - \ell(z)}{h}.$$

Note that, if  $\ell$  is Gâteaux differentiable at  $z$ , then we get

$$d\ell(z)(y) = \langle \nabla \ell(z), y \rangle \quad \text{for each } y \in Y,$$

where  $\nabla \ell$  denotes the Gâteaux derivative of  $\ell$ ; cf. [11].

The paper is organized as follows: In section 2, we set the spreading control problems in their open loop form. Then section 3 gives the basic results on feedback spreading control laws. In section 4, we state the speed functional and show some results which justify its definition. In section 5, we deal with feedback spreading control laws under speed constraints. Finally, section 6 is devoted to the optimality of these control laws.

**2. Statement of the problem.** Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded domain with sufficiently smooth boundary  $\partial\Omega$ , and set  $Q = \Omega \times (0, \infty[$ . Let  $A$  be a second order elliptic operator on  $\Omega$  given in the form

$$(2.1) \quad A \doteq - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a_0(x),$$

with the smooth functions  $a_{ij}$ ,  $a_i$ , and  $a_0$ . For convenient boundary data, it can be assumed that the operator  $-A$  stands for an unbounded densely defined linear operator which generates a  $\mathcal{C}_0$  analytic semigroup  $(S(t))_{t \geq 0}$  on  $Z = L^2(\Omega)$ ; cf. [2, 3].

We consider the semilinear parabolic control system

$$(2.2a) \quad \frac{\partial z}{\partial t} + Az = \varphi(z, v) \quad \text{in } Q$$

with initial data

$$(2.2b) \quad z(x, 0) = z_0(x) \quad \text{in } \Omega,$$

where  $z_0 \in \text{dom}(A)$  (i.e., the domain of  $A$ ) and  $\varphi$  denotes a nonlinear operator which maps  $\mathcal{S} \times V$  into  $Z$ , with  $V$  another Hilbert space and  $\mathcal{S}$  a closed subset of  $Z$ . Let  $\omega$  be a map defined as follows:

$$(2.3) \quad \omega : \mathcal{S} \subset Z \rightarrow 2^\Omega.$$

DEFINITION 2.1 (cf. [6]). *A measurable function  $\bar{v} : [0, t_1[ \rightarrow V$  is called a spreading control with respect to  $\omega$  if there exists a solution  $\bar{z}$  which satisfies*

$$(2.4a) \quad \bar{z}(t) \in \mathcal{S} \quad \text{for all } t \in [0, t_1[$$

and

$$(2.4b) \quad (\omega(\bar{z}(t)))_{0 \leq t < t_1} \text{ is nondecreasing.}$$

As an instance of the map  $\omega$ , we consider the pollution process; cf. [8, 12]. It takes place when the system which describes the *concentration of pollutant* is spreadable with respect to  $\omega$ , with

$$\omega(z) \doteq \{x \in \Omega \mid z(x) > z_{\max}\},$$

where  $z_{\max}$  is a tolerance coefficient. Let  $t_1 > 0$ , and set

$$\mathcal{V}_{t_1}^s \doteq \{v \in L^2(0, t_1, V) \mid v \text{ is a spreading control with respect to } \omega\}.$$

For each control  $v \in \mathcal{V}_{t_1}^s$ , denote by  $z(\cdot, v)$  the solution of (2.2) on the interval  $[0, t_1[$ ; then a natural way to define the speed of the generated spread  $(\omega(z(t, v)))_t$  may be

$$(2.5) \quad \text{speed}(t, v) \doteq \liminf_{h \downarrow 0} \frac{\lambda(\omega(z(t+h, v)) \setminus \omega(z(t, v)))}{h} \geq 0 \quad \text{for each } t \in [0, t_1[,$$

where  $\lambda$  stands for the Lebesgue measure on  $\Omega$ . Then, roughly, the control problems we shall consider in this paper are stated as follows:

$$P_m^+ \quad \text{Find a control } v_m^+ \in \mathcal{V}_{t_1}^s \text{ such that} \\ \text{speed}(t, v_m^+) \geq m(t) \text{ for all } t \in [0, t_1[$$

and

$$P_m^- \quad \text{Find a control } v_m^- \in \mathcal{V}_{t_1}^s \text{ such that} \\ \text{speed}(t, v_m^-) \leq m(t) \text{ for all } t \in [0, t_1[,$$

where  $m : [0, t_1[ \rightarrow \mathbb{R}^+$  stands for a measurable function. Also, we are concerned with investigating the optimal control problems

$$P_{\theta, m}^+ \quad \min \|v_s\|_{L^2(0, t_1, V)}^2 \text{ subject to} \\ v_s \text{ is a solution of } P_m^+$$

and

$$P_{\theta, m}^- \quad \min \|v_s\|_{L^2(0, t_1, V)}^2 \text{ subject to} \\ v_s \text{ is a solution of } P_m^-.$$

Note that there are two technical notes which might be taken into account:

- (i) According to [13], it should be of interest to seek feedback spreading control laws in the form

$$(2.6) \quad v_s = \psi(z) \quad \text{for each } z \in \mathcal{S}.$$

- (ii) In general, the Lebesgue measure  $\lambda$  does not provide a well-defined function  $\text{speed}(\cdot, \cdot)$ . That, a priori, depends upon the differentiability of  $\lambda \circ \omega$  in the sense of Dini; cf. [11].

**3. Preliminaries on feedback spreading control laws.** In this section, we present a summary of the main definitions and results related to the concept of feedback spreading control. Let  $\omega$  be as in (2.3).

DEFINITION 3.1 (cf. [13]). *The mapping  $\varsigma : \mathcal{S} \rightarrow V$  is said to be a feedback spreading control (fsc) law with respect to  $\omega$  if, for all initial data  $z_0$  in  $\mathcal{S}$ , there exists a solution  $\bar{z}$  which satisfies*

$$(3.1a) \quad \bar{z}(t) \in \mathcal{S} \quad \text{for all } t \in [0, t_1[$$

and

$$(3.1b) \quad \bar{v} = \varsigma(\bar{z}) \text{ is a spreading control.}$$

Next, for each couple  $(y, z) \in Z \times \mathcal{S}$ , consider the following tangential condition:

$$(3.2) \quad \begin{aligned} &\text{for all } \delta > 0, \exists 0 < h < \delta, \text{ and } \|p\| \leq \delta \text{ such that} \\ &S(h)z + h(y + p) \in \mathcal{S} \text{ and} \\ &\omega(S(h)z + h(y + p)) \supset \omega(z). \end{aligned}$$

Then define the set-valued maps

$$(3.3) \quad \mathcal{T}_\omega(z) \doteq \{y \in Z \mid (3.2) \text{ holds with } (y, z)\} \quad \text{for each } z \in \mathcal{S}$$

and

$$(3.4) \quad \mathcal{F}_\omega(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathcal{T}_\omega(z)\} \quad \text{for each } z \in \mathcal{S}.$$

Also, we need to let

$$(3.5) \quad \Sigma_\omega \doteq \{(y, z) \in \mathcal{S}^2 \mid \omega(y) \supset \omega(z)\}$$

and make the following assumption.

*Assumption 3.2.* The semigroup  $S(\cdot)$  is compact.

We are ready to present the following basic result which characterizes fsc laws.

THEOREM 3.3. *Let Assumption 3.2 hold, and let  $\varsigma : \mathcal{S} \rightarrow V$  be a measurable function. Furthermore, assume that*

- (i)  $\Sigma_\omega$  is closed,
- (ii)  $\varphi(\cdot, \varsigma(\cdot))$  is demicontinuous on  $\mathcal{S}$ .

*Then  $\varsigma$  is an fsc law with respect to  $\omega$  iff  $\varsigma$  is a selection of  $\mathcal{F}_\omega$ .*

*Proof.* See [13, Theorem 3.1].  $\square$

It should be convenient to emphasize that, in Theorem 3.3, only  $\varphi(\cdot, \varsigma(\cdot))$  is required to be demicontinuous, and there are no continuity assumptions on  $\varphi$  or  $\varsigma$ . Also, note that Assumption 3.2 is generic for parabolic systems; cf. [3, 16].

Remark 3.4. It is useful to notice that the subset  $\mathcal{T}_\omega(z)$  may be expressed in terms of contingent subsets [17], which are given by

$$T_D^A(z) = \left\{ y \in Z \mid \liminf_{h \downarrow 0} \frac{d(S(h)z + hy, D)}{h} = 0 \right\}.$$

We have, by considering (3.2),

$$\mathcal{T}_\omega(z) = T_{P_\omega(z)}^A(z) \quad \text{for each } z \in \mathcal{S},$$

where

$$(3.6) \quad P_\omega(z) = \{y \in Z \mid \omega(y) \supset \omega(z)\} \quad \text{for each } z \in \mathcal{S}.$$

In the preliminary result below, we use the following assumption.

*Assumption 3.5.*  $\Sigma_\omega$  is closed, and the map  $\omega^{-1}$  has convex values.

LEMMA 3.6.

- (i) *The map  $\mathcal{T}_\omega$  has closed values.*
- (ii) *Under Assumption 3.5, the map  $\mathcal{T}_\omega$  has convex values.*

*Proof.* Let  $z$  belong to  $\mathcal{S}$ ; then the tangential condition (3.2) yields

$$\mathcal{T}_\omega(z) = \bigcap_{\delta > 0} cl \bigcup_{h \in (0, \delta)} \frac{1}{h} [P_\omega(z) - S(h)z].$$

Then it is obvious that  $\mathcal{T}_\omega(z)$  is closed.

Regarding (ii), let  $y, \bar{y} \in \mathcal{T}_\omega(z)$ , and  $\alpha, \beta \geq 0$  such that  $\alpha + \beta = 1$ . It follows that

$$S(h)z + h(\alpha y + \beta \bar{y}) = \alpha(S(h)z + hy) + \beta(S(h)z + h\bar{y}).$$

Now it is not hard to show that Assumption 3.5 implies that the map  $P_\omega$  has closed convex values. It follows that the function

$$y \in Z \rightarrow d(y, P_\omega(z))$$

is convex, and therefore we get

$$d(S(h)z + h(\alpha y + \beta \bar{y}), P_\omega(z)) \leq \alpha d(S(h)z + hy, P_\omega(z)) + \beta d(S(h)z + h\bar{y}, P_\omega(z)).$$

Consequently, Remark 3.4 yields the result.  $\square$

**4. The speed functional.** Let  $\mu$  be a measure on  $\Omega$ . By the *speed functional*, we mean the functional defined on  $\text{graph}(\mathcal{T}_\omega)$  by

$$(4.1) \quad \theta(y, z) \doteq \liminf_{h \downarrow 0, \|p\| \rightarrow 0} \frac{\mu(\omega(S(h)z + h(y + p)) \setminus \omega(z))}{h}$$

for each  $z \in \mathcal{S}$  and  $y \in \mathcal{T}_\omega(z)$ .

Let

$$\tau_\omega \doteq \mu \circ \omega : \begin{array}{ll} \mathcal{S} & \rightarrow \mathbb{R}^+, \\ z & \rightarrow \mu(\omega(z)). \end{array}$$

Next, we prove some immediate properties which are verified by the speed functional.

PROPOSITION 4.1.

- (i)  $\theta$  is well defined and has values ranging in  $[0, \infty]$ .
- (ii) Assume  $\tau_\omega$  to be locally Lipschitz on  $\mathcal{S}$ . Let  $\bar{v}$  and  $\bar{z}$  be as in Definition 2.1, with  $\mu$  instead of  $\lambda$ ; then we have

$$(4.2) \quad \text{speed}(t, \bar{v}) = \theta(\varphi(\bar{z}(t), \bar{v}(t)), \bar{z}(t)) \quad \text{for each } t \in [0, t_1[.$$

- (iii) Suppose that  $\mathcal{S}$  and  $\tau_\omega$  are convex; then we have

$$(4.3) \quad \theta(y, z) = d\tau_\omega(z)(y - Az) \quad \text{for each } y \in \mathcal{T}_\omega(z) \text{ and } z \in \mathcal{S} \cap \text{dom}(A).$$

*Proof.* First, note that (i) is simply a consequence of (3.2) and (3.3). To show (ii), let  $\bar{v}$  and  $\bar{z}$  be as in Definition 2.1, and denote  $\bar{\varphi}(\cdot) \doteq \varphi(\bar{z}(\cdot), \bar{v}(\cdot))$ . For  $t \in [0, t_1[$ , we get

$$\bar{z}(t + h) = \bar{z}(t) + h(\bar{\varphi}(t) + p_h) \quad \text{with } p_h \rightarrow 0 \text{ when } h \rightarrow 0.$$

Then, applying formula (4.1) yields

$$\begin{aligned} \theta(\varphi(\bar{z}(t), \bar{v}(t)), \bar{z}(t)) &= \liminf_{h \downarrow 0, \|p\| \rightarrow 0} \frac{\tau_\omega(S(h)\bar{z}(t) + h(\bar{\varphi}(t) + p)) - \tau_\omega(\bar{z}(t))}{h} \\ &= \liminf_{h \downarrow 0, \|p\| \rightarrow 0} \frac{\tau_\omega(\bar{z}(t + h) + h(p - p_h)) - \tau_\omega(\bar{z}(t))}{h}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} \tau_\omega(\bar{z}(t + h) + h(p - p_h)) - \tau_\omega(\bar{z}(t)) &= \tau_\omega(\bar{z}(t + h)) - \tau_\omega(\bar{z}(t)) \\ &\quad + \tau_\omega(\bar{z}(t + h) + h(p - p_h)) - \tau_\omega(\bar{z}(t + h)). \end{aligned}$$

We use the fact that  $\tau_\omega$  is locally Lipschitz to obtain

$$\lim_{h \downarrow 0, \|p\| \rightarrow 0} \frac{\tau_\omega(\bar{z}(t + h) + h(p - p_h)) - \tau_\omega(\bar{z}(t))}{h} = 0.$$

Therefore, (ii) is proved if we refer to (2.5).

Regarding statement (iii), we first remark that, due to its convexity, the mapping  $\tau_\omega$  has a directional derivative on  $\mathcal{S}$ , and  $d\tau_\omega(z)(\cdot)$  is continuous for each  $z$ ; cf. [11]. On the other hand, we have

$$S(h)z = z - hAz + hp_h \quad \text{for each } h \geq 0, \text{ with } p_h \rightarrow 0 \text{ when } h \rightarrow 0.$$

It follows that

$$\theta(y, z) = \liminf_{h \downarrow 0, \|p\| \rightarrow 0} \frac{\tau_\omega(z + h(y - Az + p)) - \tau_\omega(z)}{h}.$$

Therefore, we have

$$\theta(y, z) = \liminf_{\|p\| \rightarrow 0} d\tau_\omega(z)(y - Az + p),$$

and consequently we obtain (4.3) thanks to the continuity of the directional derivative.  $\square$

As an important consequence, we stress that the speed functional provides a proper tool in order to measure the speed of the spread generated by a spreading control, especially when  $\tau_\omega$  has a directional derivative, in which case formula (4.3) can easily be used.

*Remark 4.2.* Note that in (iii) the assumption “ $\tau_\omega$  is convex” may be replaced by “ $\tau_\omega$  is Gâteaux differentiable.” In this case, we obtain the formula

$$(4.4) \quad \theta(y, z) = \langle \nabla \tau_\omega(z); y - Az \rangle \quad \text{for each } y \in \mathcal{T}_\omega(z) \text{ and } z \in \mathcal{S} \cap \text{dom}(A).$$

Next, we show a technical result to be used in the subsequent sections. To this end, let us consider the following assumption.

*Assumption 4.3.* For each sequence  $(z_n)_n \subset \mathcal{S}$  and  $(y_n)_n \subset Z$  such that  $y_n \in \mathcal{T}_\omega(z_n)$  for every  $n$ , we have

$$\begin{aligned} z_n \rightarrow z \text{ (strong)} \\ y_n \rightarrow y \text{ (weak)} \end{aligned} \implies y \in \mathcal{T}_\omega(z), \text{ and } \theta(y_n, z_n) \rightarrow \theta(y, z).$$

Then we can prove the following result, which studies the convexity of the mapping  $\theta(\cdot, z)$  on  $\mathcal{T}_\omega(z)$ .

**LEMMA 4.4.** *Let Assumptions 3.5 and 4.3 be satisfied; then we have the following statements:*

- (i) *If  $\tau_\omega$  is convex, then  $\theta(\cdot, z)$  is convex on  $\mathcal{T}_\omega(z)$  for each  $z \in \mathcal{S}$ .*
- (ii) *If  $\tau_\omega$  is Gâteaux differentiable, then, for each  $\alpha, \beta > 0$  with  $\alpha + \beta = 1$ , we have*

$$\theta(\alpha y + \beta \bar{y}, z) = \alpha \theta(y, z) + \beta \theta(\bar{y}, z) \quad \text{for each } z \in \mathcal{S} \text{ and } y, \bar{y} \in \mathcal{T}_\omega(z).$$

*Proof.* For  $z \in \mathcal{S} \cap \text{dom}(A)$ , by considering Proposition 4.1(iii), we can easily see that  $\theta(\cdot, z)$  is convex on  $\mathcal{T}_\omega(z)$  because  $d\tau_\omega(z)$  is such. Now let  $z \in \mathcal{S}$ ; then  $z = \lim_{n \rightarrow \infty} z_n$  for a sequence  $(z_n)_n \subset \mathcal{S} \cap \text{dom}(A)$ . Let  $\alpha, \beta \geq 0$  such that  $\alpha + \beta = 1$  and  $y, \bar{y} \in \mathcal{T}_\omega(z)$ ; then using Assumption 4.3 yields

$$\begin{aligned} \theta(\alpha y + \beta \bar{y}, z) &= \lim_{n \rightarrow \infty} \theta(\alpha y + \beta \bar{y}, z_n) \\ &\leq \alpha \lim_{n \rightarrow \infty} \theta(y, z_n) + \beta \lim_{n \rightarrow \infty} \theta(\bar{y}, z_n) \\ &\leq \alpha \theta(y, z) + \beta \theta(\bar{y}, z), \end{aligned}$$

and hence (i) is shown. Similarly, statement (ii) easily follows from Remark 4.2 and Assumption 4.3.  $\square$

**5. Feedback spreading controls with constraints on the speed.** In this section, based on the results provided by Theorem 3.3 and Proposition 4.1, we suitably restate the spreading control problems  $P_m^+$  and  $P_m^-$  of section 2 in their feedback version. Let  $\nu$  be a nonnegative measurable function on  $\mathcal{S}$ ; then problem  $P_m^+$  may read as follows:

$$\mathbb{P}_\nu^+ \quad \begin{aligned} &\text{Find an fsc law } v = \varsigma_\nu^+(z) \text{ such that} \\ &\rho(z, v) \geq \nu(z) \text{ for each } z \in \mathcal{S}. \end{aligned}$$

Also, problem  $P_m^-$  can be reformulated as

$$\mathbb{P}_\nu^- \quad \begin{aligned} &\text{Find an fsc law } v = \varsigma_\nu^-(z) \text{ such that} \\ &\rho(z, v) \leq \nu(z) \text{ for each } z \in \mathcal{S}, \end{aligned}$$

where the functional  $\rho$  is defined according to (4.2) by

$$(5.1) \quad \rho(z, v) \doteq \theta(\varphi(z, v), z) \quad \text{for each } z \in \mathcal{S} \text{ and } v \in \mathcal{F}_\omega(z),$$

and  $\mathcal{F}_\omega(\cdot)$  is as in (3.4). Now define the following maps for each  $z \in \mathcal{S}$ :

$$(5.2a) \quad \mathcal{T}_\omega^{\nu^+}(z) \doteq \{y \in \mathcal{T}_\omega(z) \mid \theta(y, z) \geq \nu(z)\}$$

and

$$(5.2b) \quad \mathcal{F}_\omega^{\nu^+}(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathcal{T}_\omega^{\nu^+}(z)\}.$$

We also need to set

$$(5.3a) \quad \mathcal{T}_\omega^{\nu^-}(z) \doteq \{y \in \mathcal{T}_\omega(z) \mid \theta(y, z) \leq \nu(z)\}$$

and

$$(5.3b) \quad \mathcal{F}_\omega^{\nu^-}(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathcal{T}_\omega^{\nu^-}(z)\}.$$

Consequently, providing that the assumptions of Theorem 3.3 are satisfied by  $\varsigma_\nu^\epsilon$  (with  $\epsilon$  denoting  $+$  or  $-$ ), the following statement holds:

$$(5.4) \quad \varsigma_\nu^\epsilon \text{ is a solution of problem } \mathbb{P}_\nu^\epsilon \iff \varsigma_\nu^\epsilon \text{ is a selection of } \mathcal{F}_\omega^{\nu^\epsilon}.$$

For both problems  $\mathbb{P}_\nu^+$  and  $\mathbb{P}_\nu^-$ , we respectively define the subsets of admissible speeds  $\nu$  as follows:

$$(5.5a) \quad \mathcal{A}_\omega^+ \doteq \{\nu : \mathcal{S} \rightarrow \mathbb{R}^+ \mid \text{for all } z \in \mathcal{S}, \exists y \in \mathcal{T}_\omega(z) \text{ such that } \theta(y, z) > \nu(z)\}$$

and

$$(5.5b) \quad \mathcal{A}_\omega^- \doteq \{\nu : \mathcal{S} \rightarrow \mathbb{R}^+ \mid \text{for all } z \in \mathcal{S}, \exists y \in \mathcal{T}_\omega(z) \text{ such that } \theta(y, z) < \nu(z)\}.$$

In order to state an existence result for problem  $\mathbb{P}_\nu^+$  for appropriate speeds  $\nu$ , we first begin by proving the following lemma, which studies the lower semicontinuity of the map  $\mathbb{P}_\nu^+$ .

LEMMA 5.1. *Let Assumptions 3.5 and 4.3 be satisfied. Furthermore, suppose that*

- (i)  $\mathcal{T}_\omega$  is lsc,
- (ii)  $\tau_\omega$  is Gâteaux differentiable,
- (iii)  $\nu \in \mathcal{A}_\omega^+$  and is upper semicontinuous.

Then the map  $\mathcal{T}_\omega^{\nu^+}$  is lsc.

*Proof.* Since  $\nu \in \mathcal{A}_\omega^+$ , it can easily be seen that  $\mathcal{T}_\omega^{\nu^+}(z) \neq \emptyset$  for each  $z \in \mathcal{S}$ . Now, to see that this map is lsc, it suffices to show that the functional

$$j : z \in \mathcal{S} \rightarrow d(y_0, \mathcal{T}_\omega^{\nu^+}(z))^2$$

is upper semicontinuous for each  $y_0 \in Y$ ; cf. [5, Lemma 4.2]. Indeed, given  $y_0 \in Y$  and  $z \in \mathcal{S}$ , we have

$$(5.6) \quad j(z) = \min_{\substack{y \in \mathcal{T}_\omega(z) \\ \nu(z) - \theta(y, z) \leq 0}} \|y_0 - y\|^2.$$

By virtue of Lemma 3.6,  $\mathcal{T}_\omega(z)$  is closed and convex. On the other hand, the function  $\nu(z) - \theta(\cdot, z)$  is continuous (by Assumption 4.3) and convex (due to Lemma 4.4). Then there is a unique  $y_+(z) \in \mathcal{T}_\omega(z)$  which solves the optimization problem (5.6). For such a problem, the fact that  $\nu \in \mathcal{A}_\omega^+$  obviously provides the Slater condition as in [11, Theorem 6.7] yields the formula

$$(5.7) \quad j(z) = \sup_{\lambda \geq 0} \inf_{y \in \mathcal{T}_\omega(z)} \{\|y_0 - y\|^2 + \lambda(\nu(z) - \theta(y, z))\} \quad \text{for each } z \in \mathcal{S}.$$

Now let  $(z_n)_n$  be a sequence in  $\mathcal{S}$  which converges to  $z$ . By condition (i) and the fact that  $y_+(z) \in \mathcal{T}_\omega(z)$ , there exists a sequence  $y_n \in \mathcal{T}_\omega(z)$  which converges to  $y_+(z)$ . It follows that

$$(5.8) \quad \inf_{y \in \mathcal{T}_\omega(z_n)} \{\|y_0 - y\|^2 + \lambda(\nu(z_n) - \theta(y, z_n))\} \leq \|y_0 - y_n\|^2 + \lambda(\nu(z_n) - \theta(y_n, z_n))$$

for each  $\lambda \geq 0$  and  $n \in \mathbb{N}$ . However, since  $\nu$  is upper semicontinuous and

$$\theta(y_n, z_n) \rightarrow \theta(y_+(z), z),$$

we get

$$\limsup_{n \rightarrow \infty} (\nu(z_n) - \theta(y_n, z_n)) \leq \nu(z) - \theta(y_+(z), z) \leq 0.$$

Consequently, by passing to the lim sup in (5.8), we obtain

$$(5.9) \quad \limsup_{n \rightarrow \infty} \inf_{y \in \mathcal{T}_\omega(z_n)} \{ \|y_0 - y\|^2 + \lambda(\nu(z_n) - \theta(y, z_n)) \} \leq \|y_0 - y_+(z)\|^2$$

for each  $\lambda \geq 0$ . Next, by writing  $j(z_n)$  by (5.7) and noting that

$$\limsup_{n \rightarrow \infty} \sup_{\lambda \geq 0} [\cdot] \leq \sup_{\lambda \geq 0} \limsup_{n \rightarrow \infty} [\cdot],$$

we get the desired inequality

$$\limsup_{n \rightarrow \infty} j(z_n) \leq j(z),$$

ending the proof of the lemma.  $\square$

Now we turn our attention to examine the lower semicontinuity of the map  $\mathcal{T}_\omega^{\nu^-}$  as defined by (5.3a). Arguing as in the proof of Lemma 5.1, we can easily show the following result.

LEMMA 5.2. *Let Assumptions 3.5 and 4.3 be satisfied. Furthermore, suppose that*

- (i)  $\mathcal{T}_\omega$  is lsc,
- (ii)  $\tau_\omega$  is convex or Gâteaux differentiable,
- (iii)  $\nu \in \mathcal{A}_\omega^-$  and is lsc.

Then the map  $\mathcal{T}_\omega^{\nu^-}$  is lsc.

Consequently, we are in a position to provide existence results for problems  $\mathbb{P}_\nu^+$  and  $\mathbb{P}_\nu^-$ . For that purpose, we need to take into consideration the following hypothesis.

Assumption 5.3. For each  $z \in \mathcal{S}$  and  $y \in \mathcal{T}_\omega(z)$ , there exists  $v \in V$  such that  $\varphi(z, v) = y$ .

PROPOSITION 5.4. *Let Assumptions 3.2 and 5.3 hold, and assume that all of the conditions of Lemma 5.1 (resp., Lemma 5.2) are satisfied. Then there exists an fsc law which solves problem  $\mathbb{P}_\nu^+$  (resp., problem  $\mathbb{P}_\nu^-$ ).*

Proof. By Lemma 5.1 (resp., Lemma 5.2), the map  $\mathcal{T}_\omega^{\nu^+}$  (resp.,  $\mathcal{T}_\omega^{\nu^-}$ ) is lsc. Moreover, due to Assumption 4.3 and Lemma 4.4, it has closed convex values. Then, thanks to Michael’s selection theorem which is stated in section 1, the map  $\mathcal{T}_\omega^{\nu^+}$  (resp.,  $\mathcal{T}_\omega^{\nu^-}$ ) admits a continuous selection  $y_+(\cdot)$  (resp.,  $y_-(\cdot)$ ). Next, we can use Assumption 5.3 to construct a mapping  $\varsigma_\nu^+ : \mathcal{S} \rightarrow V$  (resp.,  $\varsigma_\nu^-$ ) in such a manner that

$$\varphi(z, \varsigma_\nu^\epsilon(z)) = y_\epsilon(z) \quad \text{for each } z \in \mathcal{S} \text{ with } \epsilon \in \{+, -\}.$$

Consequently, by using Theorem 3.3, it follows that  $\varsigma_\nu^+$  (resp.,  $\varsigma_\nu^-$ ) stands for the desired fsc law.  $\square$

**6. Optimality of fsc laws with speed constraints.** In the same spirit of section 5, we investigate in this section the feedback versions of optimal control problems  $P_{\theta,m}^+$  and  $P_{\theta,m}^-$ , which are stated in section 2. Let  $\nu$  be a measurable function defined from  $\mathcal{S}$  with nonnegative values; then problem  $P_{\theta,m}^+$  may read as follows:

$$\mathbb{P}_{\theta,\nu}^+ \quad \begin{array}{l} \text{Find an fsc law } v = \hat{\zeta}_+(z) \text{ which solves} \\ \min \|v\|^2 \text{ subject to } \varphi(z, v) \in \mathcal{T}_\omega^{\nu^+}(z) \text{ for each } z \in \mathcal{S}. \end{array}$$

Also, problem  $P_{\theta,m}^-$  can be restated as

$$\mathbb{P}_{\theta,\nu}^- \quad \begin{array}{l} \text{Find an fsc law } v = \hat{\zeta}_-(z) \text{ which solves} \\ \min \|v\|^2 \text{ subject to } \varphi(z, v) \in \mathcal{T}_\omega^{\nu^-}(z) \text{ for each } z \in \mathcal{S}. \end{array}$$

By virtue of Theorem 3.3, the above problems can be treated through satisfying what follows.

- (a) The involved parameterized minimization problems

$$(6.1) \quad \min \|v\|^2 \text{ subject to } \varphi(z, v) \in \mathcal{T}_\omega^{\nu^\epsilon}(z) \text{ for each } z \in \mathcal{S}$$

uniquely have solutions  $\hat{\zeta}_\epsilon(\cdot)$  for each  $z \in \mathcal{S}$  and  $\epsilon \in \{+, -\}$ .

- (b) Let the mappings  $\varphi(\cdot, \hat{\zeta}_\epsilon(\cdot))$  be demicontinuous.

Next, we need to assume that (2.2a) is affine in the controls; i.e.,

$$(6.2) \quad \varphi(z, v) = B(z)v + f(z) \text{ for each } z \in \mathcal{S} \text{ and } v \in V,$$

where  $f$  and  $B$  act in  $\mathcal{S}$  and have images, respectively, in  $Z$  and  $\mathcal{L}(V, Z)$ .

First, we show a result on the optimization technique to be used in order to solve the problems (6.1).

LEMMA 6.1. *Let  $f \in Z$  and  $\mathcal{T}$  be a closed convex subset of  $Z$ . Let  $B \in \mathcal{L}(V, Z)$  be a linear operator satisfying the following condition:*

$$(6.3) \quad \|B^* \mu\|^2 \geq m \|\mu\|^2 \text{ for each } \mu \in Z \text{ with } m > 0.$$

Then the minimization problem

$$(6.4) \quad \min_{Bv+f \in \mathcal{T}} \|v\|^2$$

has a unique solution  $v_0 = -B^* \mu_0$ , where  $\mu_0$  is uniquely given by the optimality system

$$(6.5) \quad \begin{array}{l} \|B^* \mu_0\|^2 \leq \langle \mu_0, f - y \rangle \text{ for each } y \in \mathcal{T}, \\ f - BB^* \mu_0 \in \mathcal{T}. \end{array}$$

*Proof.* See the appendix.  $\square$

Now we consider the following assumption.

Assumption 6.2.

- (i) The mapping  $f : \mathcal{S} \rightarrow Z$  is continuous.
- (ii) For each sequence  $(z_n)_n \subset \mathcal{S}$ ,  $(v_n)_n \subset V$ , and  $(\mu_n)_n \subset Z$ ,

$$\begin{array}{l} z_n \rightarrow z \text{ (strong) and } v_n \rightarrow v \text{ (weak)} \implies B(z_n)v_n \rightarrow B(z)v \text{ (weak)}, \\ z_n \rightarrow z \text{ (strong) and } \mu_n \rightarrow \mu \text{ (weak)} \implies B^*(z_n)\mu_n \rightarrow B^*(z)\mu \text{ (weak)}. \end{array}$$

(iii) For each  $z \in \mathcal{S}$ , the operator  $B^*(z)$  satisfies the coercivity condition which is required in (6.3),

$$(6.6) \quad \|B^*(z)\mu\|^2 \geq m_z \|\mu\|^2 \quad \text{for each } \mu \in Z,$$

where the coefficient  $m_z > 0$  is such that, for each  $\alpha > 0$ , there exists  $M > 0$  such that  $m_z > M$  for each  $z \in \mathcal{S}$ ,  $\|z\| < \alpha$ .

Then we are ready to examine problem  $\mathbb{P}_{\theta, \nu}^+$ .

**THEOREM 6.3.** *Let Assumptions 3.2, 3.5, and 6.2 be satisfied. In addition, suppose that*

- (i) *the map  $\mathcal{T}_\omega$  is lsc,*
- (ii)  *$\tau_\omega$  is Gâteaux differentiable,*
- (iii)  *$\nu \in \mathcal{A}_\omega^+$  and is upper semicontinuous.*

*Then the mapping  $\hat{\zeta}_+(\cdot)$  of (6.1) stands for the unique solution of problem  $\mathbb{P}_{\theta, \nu}^+$ .*

*Proof.* By Lemma 4.4, the map  $\mathcal{T}_\omega$  has closed convex values. Then the map  $\mathcal{T}_\omega^{\nu^+}$  of (5.2a) also has closed convex values. This results, respectively, from Assumption 4.3 and Lemma 4.4 (ii). Therefore, all conditions of Lemma 6.1 are satisfied for each  $z \in \mathcal{S}$  with  $B \doteq B(z)$ ,  $f \doteq f(z)$ ,  $\mathcal{T} \doteq \mathcal{T}_\omega^{\nu^+}(z)$ , and the coercivity condition (6.6). Hence the minimization problem (6.1) has  $\hat{\zeta}_+(z)$  as a unique solution for each  $z \in \mathcal{S}$ . In addition, by using Lemma 6.1, we have

$$(6.7) \quad \hat{\zeta}_+(z) = -B^*(z)\mu_0(z) \quad \text{for each } z \in \mathcal{S},$$

where  $\mu_0(z) \doteq \mu_0$  is uniquely determined by

$$(6.8) \quad \begin{aligned} \|B^*(z)\mu_0\|^2 &\leq \langle \mu_0, f(z) - y \rangle \quad \text{for each } y \in \mathcal{T}_\omega^{\nu^+}(z) \text{ and } z \in \mathcal{S}, \\ f(z) - B(z)B^*(z)\mu_0 &\in \mathcal{T}_\omega^{\nu^+}(z). \end{aligned}$$

Now it remains to show that  $\hat{\zeta}_+(\cdot)$  stands for an fsc law. According to (b) above, this holds if the mapping

$$\phi_s \doteq f + B(\cdot)\hat{\zeta}_+ = f - B(\cdot)B^*(\cdot)\mu_0(\cdot)$$

is demicontinuous.

Indeed, let  $(z_n)_n$  be a sequence with (strong) limit  $z \in \mathcal{S}$  and  $y \in \mathcal{T}_\omega^{\nu^+}(z)$ . Due to Lemma 5.1, the map  $\mathcal{T}_\omega^{\nu^+}$  is lsc; then there exists a sequence  $(y_n)_n$  which converges to  $y$  and satisfies

$$y_n \in \mathcal{T}_\omega^{\nu^+}(z_n) \quad \text{for each } n.$$

Therefore, by (6.8), we have

$$\|B^*(z_n)\mu_0(z_n)\|^2 \leq \langle \mu_0(z_n), f(z_n) - y_n \rangle \quad \text{for each } n.$$

Consequently, since the sequence  $(f(z_n))_n$  is bounded (due to Assumption 6.2 (i)), it follows that the sequence  $(\mu_0(z_n))_n$  is bounded too. It therefore has a subsequence  $(\mu_0(z_k))_k$  which is weakly convergent to  $\bar{\mu}_0 \in Z$ .

Now, since  $\langle \mu_0(z_k); f(z_k) - y_k \rangle \rightarrow \langle \bar{\mu}_0; f(z) - y \rangle$  (because  $f(z_k) - y_k \rightarrow f(z) - y$  strongly and  $\mu_0(z_k) \rightarrow \bar{\mu}_0$  weakly), we get by passing to the lim inf in the last inequality

$$\liminf \|B^*(z_k)\mu_0(z_k)\|^2 \leq \langle \bar{\mu}_0; f(z) - y \rangle.$$

Therefore, due to Assumption 6.2 (ii), it follows that

$$(6.9) \quad \|B^*(z)\bar{\mu}_0\|^2 \leq \liminf \|B^*(z_k)\mu_0(z_k)\|^2 \leq \langle \bar{\mu}_0; f(z) - y \rangle$$

for every  $y \in \mathcal{T}_\omega^{\nu^+}(z)$ . By using Assumption 6.2 (ii), we get

$$\phi_s(z_k) = f(z_k) - B(z_k)B^*(z_k)\mu_0(z_k) \xrightarrow{we} f(z) - B(z)B^*(z)\bar{\mu}_0.$$

This implies, due to Assumption 3.5 and the upper semicontinuity of  $\nu$ , that

$$f(z) - B(z)B^*(z)\bar{\mu}_0 \in \mathcal{T}_\omega^{\nu^+}(z).$$

Therefore, by (6.9),  $\bar{\mu}_0$  satisfies the optimality system (6.8), and then we obtain, by uniqueness,

$$\bar{\mu}_0 = \mu_0(z) \text{ and } f(z) - B(z)B^*(z)\bar{\mu}_0 = \phi_s(z).$$

Consequently, the sequences  $(\mu_0(z_n))_n$  and  $(\phi_s(z_n))_n$  are, respectively, weakly convergent to  $\mu_0(z)$  and  $\phi_s(z)$ . Thus, as desired, the mapping  $\phi_s$  is demicontinuous on the subset  $\mathcal{S}$ .  $\square$

*Remark 6.4.* From the proof of Theorem 6.3, it follows by Assumption 6.2 that the minimal fsc law

$$\hat{c}_\cdot = -B^*(\cdot)\mu_0(\cdot)$$

is also demicontinuous.

*Remark 6.5.* The proof of Lemma 6.1 in the appendix is informative on the technique to use in order to compute  $\hat{c}_\cdot$ . In fact, we can use the optimality system (A.3), from which a sequence of suboptimal fsc laws can be derived by successive approximation.

Similarly, we can follow the same approach to examine problem  $\mathbb{P}_{\theta,\nu}^-$ .

**THEOREM 6.6.** *Let Assumptions 3.2, 3.5, 5.3, and 6.2 be satisfied. In addition, suppose that*

- (i) *the map  $\mathcal{T}_\omega$  is lsc,*
- (ii)  *$\tau_\omega$  is convex or Gâteaux differentiable,*
- (iii)  *$\nu \in \mathcal{A}_\omega^-$  and is lsc.*

*Then the mapping  $\hat{c}_\cdot(\cdot)$  of (6.1) stands for the unique solution of problem  $\mathbb{P}_{\theta,\nu}^-$ .*

Also, note that Remarks 6.4 and 6.5 remain valid regarding problem  $\mathbb{P}_{\theta,\nu}^-$ .

**Appendix. Proof of Lemma 6.1.** Since  $\mathcal{C} = \{v \in V \mid Bv + f \in \mathcal{T}\}$  is a nonempty closed convex subset in  $V$ , (6.4) has a unique solution which is  $v_0 = \pi_{\mathcal{C}}(0)$ . Now we can use a saddle point method to compute  $v_0$ . Define the Lagrangian functional (cf. [10, 14])

$$L(v, y, \mu) = \frac{1}{2}\|v\|^2 + \langle Bv + f - y; \mu \rangle \quad \text{for each } v \in V, y \in \mathcal{T}, \mu \in Z.$$

In fact, it can be easily shown that, if  $(u_0, y_0, \mu_0)$  is a saddle point for  $L$ , i.e.,

$$\max_{\mu \in Z} L(u_0, y_0, \mu) = L(u_0, y_0, \mu_0) = \min_{v \in V, y \in \mathcal{T}} L(v, y, \mu_0),$$

then  $u_0$  is a solution of (6.4), and, by uniqueness,  $u_0 = v_0$ . Now, since both  $L$  and  $\mathcal{T}$  are convex, the saddle point  $(v_0, y_0, \mu_0)$  is characterized by

$$\begin{aligned} \frac{\partial L}{\partial v}(v_0, y_0, \mu_0) &= 0, \\ \left\langle \frac{\partial L}{\partial y}(v_0, y_0, \mu_0); y - y_0 \right\rangle &\geq 0 \quad \text{for each } y \in \mathcal{T}, \\ \frac{\partial L}{\partial \mu}(v_0, y_0, \mu_0) &= 0 \end{aligned}$$

so that we have

$$\begin{aligned} v_0 + B^* \mu_0 &= 0, \\ \langle \mu_0; y - y_0 \rangle &\leq 0 \quad \text{for each } y \in \mathcal{T}, \\ Bv_0 + f &= y_0 \in \mathcal{T}. \end{aligned}$$

Therefore, in an equivalent way, we get  $v_0 = -B^* \mu_0$ , where  $\mu_0$  is uniquely given by the system

$$(A.1) \quad \begin{aligned} -BB^* \mu_0 + f &= y_0, \\ \langle \mu_0; y - y_0 \rangle &\leq 0 \quad \text{for each } y \in \mathcal{T}, \\ y_0 &\in \mathcal{T}, \end{aligned}$$

which is equivalent to

$$(A.2) \quad \begin{aligned} \|B^* \mu_0\|^2 &\leq \langle \mu_0, f - y \rangle \quad \text{for each } y \in \mathcal{T}, \\ f - BB^* \mu_0 &\in \mathcal{T}. \end{aligned}$$

Now it remains to show that such a  $\mu_0$  exists. In fact, by multiplying by  $\rho > 0$  in (A.1) and using the operator of best approximation  $\pi_{\mathcal{T}}$ , we obtain the equivalent system

$$(A.3) \quad \begin{aligned} v_0 &= B^* R^{-1}(y_0 - f), \\ y_0 &= \pi_{\mathcal{T}}[(1 - \rho R^{-1})y_0 + \rho R^{-1}f] \end{aligned}$$

for some  $\rho > 0$ , where the operator  $R = BB^*$ . Then we are led to seek a fixed point of the mapping

$$\Theta_{\rho} : \begin{aligned} \mathcal{T} &\rightarrow \mathcal{T} \\ y &\rightarrow \pi_{\mathcal{T}}[(1 - \rho R^{-1})y + \rho R^{-1}f]. \end{aligned}$$

Indeed, we have

$$\begin{aligned} \|\Theta_{\rho}(y) - \Theta_{\rho}(\bar{y})\|^2 &\leq \|(1 - \rho R^{-1})e\|^2 \\ &= \|e\|^2 - 2\rho \langle R^{-1}e; e \rangle + \rho^2 \|R^{-1}e\|^2, \end{aligned}$$

where  $y, \bar{y} \in \mathcal{T}$ , and  $e = y - \bar{y}$ .

Since the operator  $R^{-1}$  is coercive, we have, for some  $m' > 0$ ,

$$\langle R^{-1}y; y \rangle \geq m' \|y\|^2 \quad \text{for each } y \in Z.$$

It follows that

$$\|\Theta_{\rho}(y) - \Theta_{\rho}(\bar{y})\|^2 \leq (1 - 2\rho m' + \rho^2 \|R^{-1}\|^2) \|y - \bar{y}\|^2.$$

Therefore,  $\Theta_{\rho}$  is a contraction for  $\rho < 2m' / \|R^{-1}\|^2$ , and thereby it has a unique fixed point  $y_0$ , which belongs to  $\mathcal{T}$ . This ends the proof of Lemma 6.1.

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